Subject: Discrete Mathematics
Topic: partial order relations
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PARTIAL ORDER RELATIONS

Partial Order Relations

A relation R on a set A is called a partial order relation if it satisfies the following three properties:

1. Relation R is Reflexive, i.e. aRa $\forall$ a$\in$A.
2. Relation R is Antisymmetric, i.e., aRb and bRa $\Rightarrow$a = b.
3. Relation R is transitive, i.e., aRb and bRc $\Rightarrow$aRc.

Example1: Show whether the relation (x, y) $\in$ R, if, x $\geq$ y defined on the set of +ve integers is a partial order relation.

Solution: Consider the set A = {1, 2, 3, 4} containing four +ve integers. Find the relation for this set such as R = {(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3), (1, 1), (2, 2), (3, 3), (4, 4)}.

Reflexive: The relation is reflexive as for every a ∈ A. (a, a) ∈ R, i.e. (1, 1), (2, 2), (3, 3), (4, 4) ∈ R.

Antisymmetric: The relation is antisymmetric as whenever (a, b) and (b, a) ∈ R, we have a = b.

Transitive: The relation is transitive as whenever (a, b) and (b, c) ∈ R, we have (a, c) ∈ R.

Example: (4, 2) ∈ R and (2, 1) ∈ R, implies (4, 1) ∈ R.

As the relation is reflexive, antisymmetric and transitive. Hence, it is a partial order relation.

Example2: Show that the relation 'Divides' defined on N is a partial order relation.

Solution:

Reflexive: We have a divides a, ∀ a∈N. Therefore, relation 'Divides' is reflexive.

Antisymmetric: Let a, b, c ∈N, such that a divides b. It implies b divides a iff a = b. So, the relation is antisymmetric.

Transitive: Let a, b, c ∈N, such that a divides b and b divides c.

Then a divides c. Hence the relation is transitive. Thus, the relation being reflexive, antisymmetric and transitive, the relation 'divides' is a partial order relation.

Example3: (a) The relation $\subseteq$ of a set of inclusion is a partial ordering or any collection of sets since set inclusion has three desired properties:

1. $A \subseteq A$ for any set A.
2. If $A \subseteq B$ and $B \subseteq A$ then $B = A$.
3. If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$

(b) The relation $\leq$ on the set R of real no that is Reflexive, Antisymmetric and transitive.

(c) Relation $\leq$ is a Partial Order Relation.

Partial Order Set (POSET):

The set A together with a partial order relation R on the set A and is denoted by (A, R) is called a partial orders set or POSET

Hasse diagram

A **Hasse diagram** is a *graphical representation* of the relation of elements of a **partially ordered set (poset)** with an implied *upward orientation*. A point is drawn for each element of the partially ordered set (poset) and joined with the line segment according to the following rules:

- If $p<q$ in the poset, then the point corresponding to *p appearslower* in the drawing than the point corresponding to q.
- The two points p and q will be joined by line segment *iff p is related to q*.

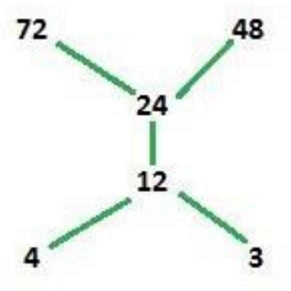
To draw a Hasse diagram, provided set must be a poset.

A poset or partially ordered set A is a pair, (B,) of a set B whose elements are called the vertices of A and obeys following rules:

1. Reflexivity → p p p B
2. Anti-symmetric → p q and q p iff p=q
3. Transitivity → if p q and q r then p r

Example-1: Draw Hasse diagram for ({3, 4, 12, 24, 48, 72}, /)

According to above given question first, we have to find the poset for the divisibility.
Let the set is A.
A={(3 12), (3 24), (3 48), (3 72), (4 12), (4 24), (4 48), (4 72), (12 24), (12 48), (12 72), (24 48), (24 72)}
So, now the Hasse diagram will be:



In above diagram, 3 and 4 are at same level because they are not related to each other and they are smaller than other elements in the set. The next succeeding element for 3 and 4 is 12 i.e, 12 is divisible by both 3 and 4. Then 24 is divisible by 3, 4 and 12. Hence, it is placed above 12. 24 divides both 48 and 72 but 48 does not divide 72. Hence 48 and 72 are not joined.

We can see transitivity in our diagram as the level is increasing.