15-251: Great theoretical ideas in Computer Science

Carnegie Mellon University

Notes on group theory

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Groups

We begin our study of algebraic structures by investigating sets associated with single operations that satisfy certain reasonable axioms; that is, we want to define an operation on a set in a way that will generalize such familiar structures as the integers $\mathbb{Z}$ together with the single operation of addition, or invertible 2×2 matrices together with the single operation of matrix multiplication. The integers and the 2×2 matrices, together with their respective single operations, are examples of algebraic structures known as groups.

The theory of groups occupies a central position in mathematics. Modern group theory arose from an attempt to find the roots of a polynomial in terms of its coefficients. Groups now play a central role in such areas as coding theory, counting, and the study of symmetries; many areas of biology, chemistry, and physics have benefited from group theory.

1.1 Integer Equivalence Classes and Symmetries

Let us now investigate some mathematical structures that can be viewed as sets with single operations.

The Integers mod n

The integers mod n have become indispensable in the theory and applications of algebra. In mathematics they are used in cryptography, coding theory, and the detection of errors in identification codes.

We have already seen that two integers a and b are equivalent mod n if n divides $a - b$. The integers mod n also partition $\mathbb{Z}$ into n different equivalence classes; we will denote the set of these equivalence classes by $\mathbb{Z}_n$. Consider the integers modulo 12 and the corresponding partition of the integers:

$$\begin{aligned}
[0] &= \{\ldots, -12, 0, 12, 24, \ldots\}, \\
[1] &= \{\ldots, -11, 1, 13, 25, \ldots\}, \\
&\vdots \\
[11] &= \{\ldots, -1, 11, 23, 35, \ldots\}.
\end{aligned}$$

When no confusion can arise, we will use $0, 1, \ldots, 11$ to indicate the equivalence classes $[0], [1], \ldots, [11]$ respectively. We can do arithmetic on $\mathbb{Z}_n$. For two integers a and b, define addition modulo n to be

$(a+b) \pmod{n}$; that is, the remainder when $a+b$ is divided by n. Similarly, multiplication modulo n is defined as $(ab) \pmod{n}$, the remainder when ab is divided by n.

Table 1.1. Multiplication table for $\mathbb{Z}_8$

$\cdot$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	6
3	0	3	6	1	4	7	2	5
4	0	4	0	4	0	4	0	4
5	0	5	2	7	4	1	6	3
6	0	6	4	2	0	6	4	2
7	0	7	6	5	4	3	2	1

Example 1. The following examples illustrate integer arithmetic modulo n:

$$7 + 4 \equiv 1 \pmod{5} \qquad 7 \cdot 3 \equiv 1 \pmod{5}$$

$$3 + 5 \equiv 0 \pmod{8} \qquad 3 \cdot 5 \equiv 7 \pmod{8}$$

$$3 + 4 \equiv 7 \pmod{12} \qquad 3 \cdot 4 \equiv 0 \pmod{12}.$$

In particular, notice that it is possible that the product of two nonzero numbers modulo n can be equivalent to 0 modulo n. ■

Example 2. Most, but not all, of the usual laws of arithmetic hold for addition and multiplication in $\mathbb{Z}_n$. For instance, it is not necessarily true that there is a multiplicative inverse. Consider the multiplication table for $\mathbb{Z}_8$ in Table 1.1. Notice that 2, 4, and 6 do not have multiplicative inverses; that is, for $n = 2$, 4, or 6, there is no integer k such that $kn \equiv 1 \pmod{8}$. ■

Proposition 1.1 *Let $\mathbb{Z}_n$ be the set of equivalence classes of the integers mod n and $a, b, c \in \mathbb{Z}_n$.*

1. *Addition and multiplication are commutative:*

$$a + b \equiv b + a \pmod{n}$$
$$ab \equiv ba \pmod{n}.$$

2. *Addition and multiplication are associative:*

$$(a + b) + c \equiv a + (b + c) \pmod{n}$$
$$(ab)c \equiv a(bc) \pmod{n}.$$

3. *There are both an additive and a multiplicative identity:*

$$a + 0 \equiv a \pmod{n}$$
$$a \cdot 1 \equiv a \pmod{n}.$$

4. *Multiplication distributes over addition:*

$$a(b + c) \equiv ab + ac \pmod{n}.$$

5. *For every integer a there is an additive inverse $-a$:*

$$a + (-a) \equiv 0 \pmod{n}.$$

6. *Let a be a nonzero integer. Then $\gcd(a, n) = 1$ if and only if there exists a multiplicative inverse b for $a \pmod{n}$; that is, a nonzero integer b such that*

$$ab \equiv 1 \pmod{n}.$$

Proof. We will prove (1) and (6) and leave the remaining properties to be proven in the exercises.

(1) Addition and multiplication are commutative modulo n since the remainder of $a + b$ divided by n is the same as the remainder of $b + a$ divided by n.

(6) Suppose that $\gcd(a, n) = 1$. Then there exist integers r and s such that $ar + ns = 1$. Since $ns = 1 - ar$, $ra \equiv 1 \pmod{n}$. Letting b be the equivalence class of r, $ab \equiv 1 \pmod{n}$.

Conversely, suppose that there exists a b such that $ab \equiv 1 \pmod{n}$. Then n divides $ab - 1$, so there is an integer k such that $ab - nk = 1$. Let $d = \gcd(a, n)$. Since d divides $ab - nk$, d must also divide 1; hence, $d = 1$. □

Symmetries

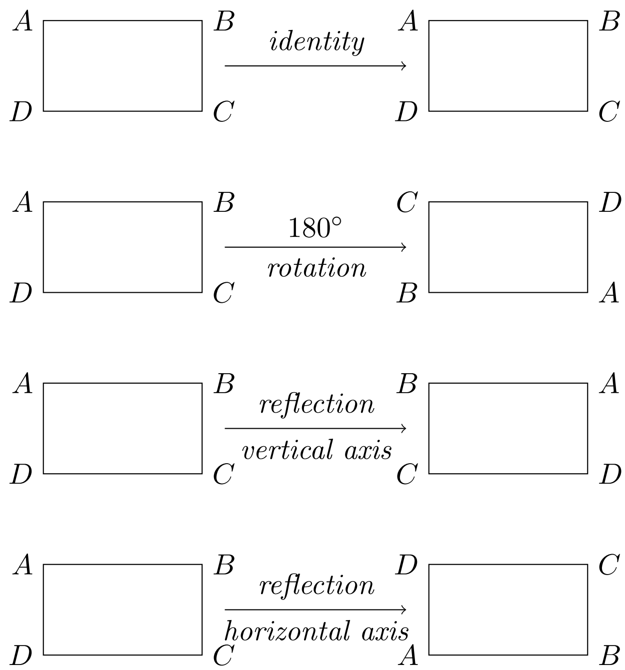
A ***symmetry*** of a geometric figure is a rearrangement of the figure preserving the arrangement of its sides and vertices as well as its distances and angles. A map from the plane to itself preserving the symmetry of an object is called a ***rigid motion***. For example, if we look at the rectangle in Figure 1.1, it is easy to see that a rotation of 180° or 360° returns a rectangle in the plane with the same orientation as the original rectangle and the same relationship among the vertices. A reflection of the rectangle across either the vertical axis or the horizontal axis can also be seen to be a symmetry. However, a 90° rotation in either direction cannot be a symmetry unless the rectangle is a square.

Let us find the symmetries of the equilateral triangle $\triangle ABC$. To find a symmetry of $\triangle ABC$, we must first examine the permutations of the vertices A, B, and C and then ask if a permutation extends to a symmetry of the triangle. Recall that a ***permutation*** of a set S is a one-to-one and onto map $\pi : S \to S$. The three vertices have $3! = 6$ permutations, so the triangle has at most six symmetries. To see that there are six permutations, observe there are three different possibilities for the first vertex, and two for the second, and the remaining vertex is determined by the placement of the first two. So we have $3 \cdot 2 \cdot 1 = 3! = 6$ different arrangements. To denote the permutation of the vertices of an equilateral triangle that sends A to B, B to C, and C to A, we write the array

$$\begin{pmatrix} A & B & C \\ B & C & A \end{pmatrix}.$$

Notice that this particular permutation corresponds to the rigid motion of rotating the triangle by 120° in a clockwise direction. In fact, every permutation gives rise to a symmetry of the triangle. All of these symmetries are shown in Figure 1.2.

Figure 1.1. Rigid motions of a rectangle



A natural question to ask is what happens if one motion of the triangle $\triangle ABC$ is followed by another. Which symmetry is $\mu_1\rho_1$; that is, what happens when we do the permutation ρ_1 and then the permutation μ_1? *Remember that we are composing functions here. Although we usually multiply left to right, we compose functions right to left.* We have

$$\begin{aligned}(\mu_1\rho_1)(A) &= \mu_1(\rho_1(A)) = \mu_1(B) = C\\ (\mu_1\rho_1)(B) &= \mu_1(\rho_1(B)) = \mu_1(C) = B\\ (\mu_1\rho_1)(C) &= \mu_1(\rho_1(C)) = \mu_1(A) = A.\end{aligned}$$

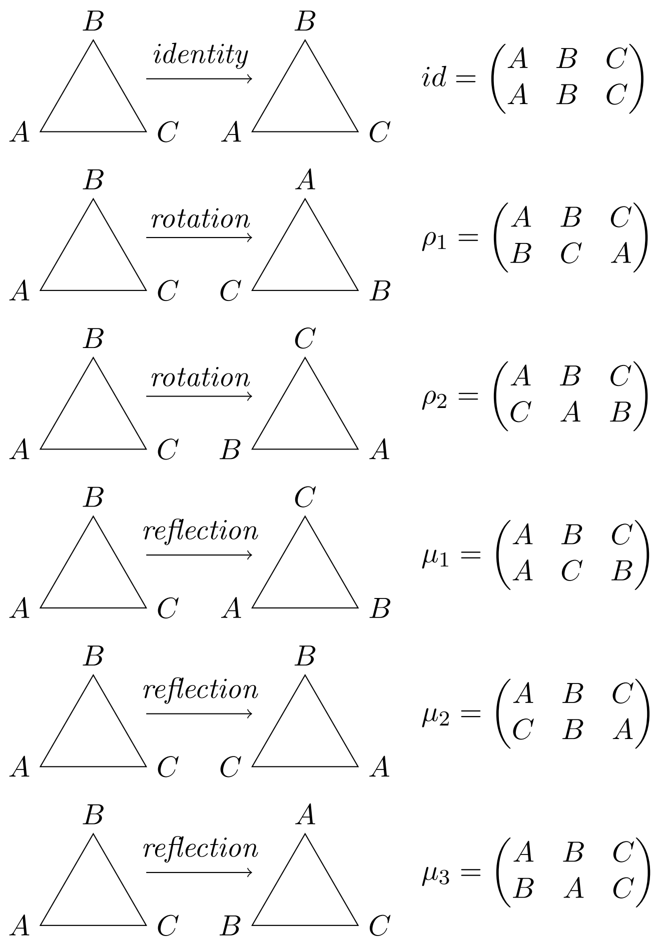
This is the same symmetry as μ_2. Suppose we do these motions in the opposite order, ρ_1 then μ_1. It is easy to determine that this is the same as the symmetry μ_3; hence, $\rho_1\mu_1 \neq \mu_1\rho_1$. A multiplication table for the symmetries of an equilateral triangle $\triangle ABC$ is given in Table 1.2.

Notice that in the multiplication table for the symmetries of an equilateral triangle, for every motion of the triangle α there is another motion α' such that $\alpha\alpha' = id$; that is, for every motion there is another motion that takes the triangle back to its original orientation.

1.2 Definitions and Examples

The integers mod n and the symmetries of a triangle or a rectangle are both examples of groups. A ***binary operation*** or ***law of composition*** on a set G is a function $G \times G \to G$ that assigns to each pair $(a, b) \in G \times G$ a unique element $a \circ b$, or ab in G, called the composition of a and b. A ***group*** $(G, \circ)$ is a set G together with a law of composition $(a, b) \mapsto a \circ b$ that satisfies the following axioms.

Figure 1.2. Symmetries of a triangle



- The law of composition is ***associative***. That is,
$$(a \circ b) \circ c = a \circ (b \circ c)$$
for $a, b, c \in G$.
- There exists an element $e \in G$, called the ***identity element***, such that for any element $a \in G$
$$e \circ a = a \circ e = a.$$
- For each element $a \in G$, there exists an ***inverse element*** in G, denoted by a^{-1}, such that
$$a \circ a^{-1} = a^{-1} \circ a = e.$$

A group G with the property that $a \circ b = b \circ a$ for all $a, b \in G$ is called ***abelian*** or ***commutative***. Groups not satisfying this property are said to be ***nonabelian*** or ***noncommutative***.

Example 3. The integers $\mathbb{Z} = \{\ldots, -1, 0, 1, 2, \ldots\}$ form a group under the operation of addition. The binary operation on two integers $m, n \in \mathbb{Z}$ is just their sum. Since the integers under addition

Table 1.2. Symmetries of an equilateral triangle

$\circ$	id	ρ_1	ρ_2	μ_1	μ_2	μ_3
id	id	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_1	ρ_1	ρ_2	id	μ_3	μ_1	μ_2
ρ_2	ρ_2	id	ρ_1	μ_2	μ_3	μ_1
μ_1	μ_1	μ_2	μ_3	id	ρ_1	ρ_2
μ_2	μ_2	μ_3	μ_1	ρ_2	id	ρ_1
μ_3	μ_3	μ_1	μ_2	ρ_1	ρ_2	id

already have a well-established notation, we will use the operator $+$ instead of $\circ$; that is, we shall write $m + n$ instead of $m \circ n$. The identity is 0, and the inverse of $n \in \mathbb{Z}$ is written as $-n$ instead of n^{-1}. Notice that the integers under addition have the additional property that $m + n = n + m$ and are therefore an abelian group. ■

Most of the time we will write ab instead of $a \circ b$; however, if the group already has a natural operation such as addition in the integers, we will use that operation. That is, if we are adding two integers, we still write $m + n$, $-n$ for the inverse, and 0 for the identity as usual. We also write $m - n$ instead of $m + (-n)$.

Table 1.3. Cayley table for $(\mathbb{Z}_5, +)$

$+$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

It is often convenient to describe a group in terms of an addition or multiplication table. Such a table is called a ***Cayley table***.

Example 4. The integers mod n form a group under addition modulo n. Consider $\mathbb{Z}_5$, consisting of the equivalence classes of the integers 0, 1, 2, 3, and 4. We define the group operation on $\mathbb{Z}_5$ by modular addition. We write the binary operation on the group additively; that is, we write $m + n$. The element 0 is the identity of the group and each element in $\mathbb{Z}_5$ has an inverse. For instance, $2 + 3 = 3 + 2 = 0$. Table 1.3 is a Cayley table for $\mathbb{Z}_5$. By Proposition 1.1, $\mathbb{Z}_n = \{0, 1, \ldots, n - 1\}$ is a group under the binary operation of addition mod n. ■

Example 5. Not every set with a binary operation is a group. For example, if we let modular multiplication be the binary operation on $\mathbb{Z}_n$, then $\mathbb{Z}_n$ fails to be a group. The element 1 acts as a group identity since $1 \cdot k = k \cdot 1 = k$ for any $k \in \mathbb{Z}_n$; however, a multiplicative inverse for 0 does not exist since $0 \cdot k = k \cdot 0 = 0$ for every k in $\mathbb{Z}_n$. Even if we consider the set $\mathbb{Z}_n \setminus \{0\}$, we still may not have a group. For instance, let $2 \in \mathbb{Z}_6$. Then 2 has no multiplicative inverse since

$$\begin{aligned} 0 \cdot 2 &= 0 & 1 \cdot 2 &= 2 \\ 2 \cdot 2 &= 4 & 3 \cdot 2 &= 0 \\ 4 \cdot 2 &= 2 & 5 \cdot 2 &= 4. \end{aligned}$$

By Proposition 1.1, every nonzero k does have an inverse in $\mathbb{Z}_n$ if k is relatively prime to n. Denote the set of all such nonzero elements in $\mathbb{Z}_n$ by $U(n)$. Then $U(n)$ is a group called the ***group of units*** of $\mathbb{Z}_n$. Table 1.4 is a Cayley table for the group $U(8)$. ■

Table 1.4. Multiplication table for $U(8)$

$\cdot$	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

Example 6. The symmetries of an equilateral triangle described in Section 1.1 form a nonabelian group. As we observed, it is not necessarily true that $\alpha\beta = \beta\alpha$ for two symmetries α and β. Using Table 1.2, which is a Cayley table for this group, we can easily check that the symmetries of an equilateral triangle are indeed a group. We will denote this group by either S_3 or D_3, for reasons that will be explained later. ■

Example 7. We use $\mathbb{M}_2(\mathbb{R})$ to denote the set of all 2×2 matrices. Let $GL_2(\mathbb{R})$ be the subset of $\mathbb{M}_2(\mathbb{R})$ consisting of invertible matrices; that is, a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is in $GL_2(\mathbb{R})$ if there exists a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$, where I is the 2×2 identity matrix. For A to have an inverse is equivalent to requiring that the determinant of A be nonzero; that is, $\det A = ad - bc \neq 0$. The set of invertible matrices forms a group called the ***general linear group***. The identity of the group is the identity matrix

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The inverse of $A \in GL_2(\mathbb{R})$ is

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

The product of two invertible matrices is again invertible. Matrix multiplication is associative, satisfying the other group axiom. For matrices it is not true in general that $AB = BA$; hence, $GL_2(\mathbb{R})$ is another example of a nonabelian group. ■

Example 8. Let

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$J = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad K = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

where $i^2 = -1$. Then the relations $I^2 = J^2 = K^2 = -1$, $IJ = K$, $JK = I$, $KI = J$, $JI = -K$, $KJ = -I$, and $IK = -J$ hold. The set $Q_8 = \{\pm 1, \pm I, \pm J, \pm K\}$ is a group called the ***quaternion group***. Notice that Q_8 is noncommutative. ■

Example 9. Let $\mathbb{C}^*$ be the set of nonzero complex numbers. Under the operation of multiplication $\mathbb{C}^*$ forms a group. The identity is 1. If $z = a + bi$ is a nonzero complex number, then

$$z^{-1} = \frac{a - bi}{a^2 + b^2}$$

is the inverse of z. It is easy to see that the remaining group axioms hold. ■

A group is ***finite***, or has ***finite order***, if it contains a finite number of elements; otherwise, the group is said to be ***infinite*** or to have ***infinite order***. The ***order*** of a finite group is the number of elements that it contains. If G is a group containing n elements, we write $|G| = n$. The group $\mathbb{Z}_5$ is a finite group of order 5; the integers $\mathbb{Z}$ form an infinite group under addition, and we sometimes write $|\mathbb{Z}| = \infty$.

Basic Properties of Groups

Proposition 1.2 *The identity element in a group G is unique; that is, there exists only one element $e \in G$ such that $eg = ge = g$ for all $g \in G$.*

Proof. Suppose that e and e' are both identities in G. Then $eg = ge = g$ and $e'g = ge' = g$ for all $g \in G$. We need to show that $e = e'$. If we think of e as the identity, then $ee' = e'$; but if e' is the identity, then $ee' = e$. Combining these two equations, we have $e = ee' = e'$. □

Inverses in a group are also unique. If g' and g'' are both inverses of an element g in a group G, then $gg' = g'g = e$ and $gg'' = g''g = e$. We want to show that $g' = g''$, but $g' = g'e = g'(gg'') = (g'g)g'' = eg'' = g''$. We summarize this fact in the following proposition.

Proposition 1.3 *If g is any element in a group G, then the inverse of g, g^{-1}, is unique.*

Proposition 1.4 *Let G be a group. If $a, b \in G$, then $(ab)^{-1} = b^{-1}a^{-1}$.*

Proof. Let $a, b \in G$. Then $abb^{-1}a^{-1} = aea^{-1} = aa^{-1} = e$. Similarly, $b^{-1}a^{-1}ab = e$. But by the previous proposition, inverses are unique; hence, $(ab)^{-1} = b^{-1}a^{-1}$. □

Proposition 1.5 *Let G be a group. For any $a \in G$, $(a^{-1})^{-1} = a$.*

Proof. Observe that $a^{-1}(a^{-1})^{-1} = e$. Consequently, multiplying both sides of this equation by a, we have

$$(a^{-1})^{-1} = e(a^{-1})^{-1} = aa^{-1}(a^{-1})^{-1} = ae = a.$$

□

It makes sense to write equations with group elements and group operations. If a and b are two elements in a group G, does there exist an element $x \in G$ such that $ax = b$? If such an x does exist, is it unique? The following proposition answers both of these questions positively.

Proposition 1.6 *Let G be a group and a and b be any two elements in G. Then the equations $ax = b$ and $xa = b$ have unique solutions in G.*

Proof. Suppose that $ax = b$. We must show that such an x exists. Multiplying both sides of $ax = b$ by a^{-1}, we have $x = ex = a^{-1}ax = a^{-1}b$.

To show uniqueness, suppose that x_1 and x_2 are both solutions of $ax = b$; then $ax_1 = b = ax_2$. So $x_1 = a^{-1}ax_1 = a^{-1}ax_2 = x_2$. The proof for the existence and uniqueness of the solution of $xa = b$ is similar. □

Proposition 1.7 *If G is a group and $a, b, c \in G$, then $ba = ca$ implies $b = c$ and $ab = ac$ implies $b = c$.*

This proposition tells us that the ***right and left cancellation laws*** are true in groups. We leave the proof as an exercise.

We can use exponential notation for groups just as we do in ordinary algebra. If G is a group and $g \in G$, then we define $g^0 = e$. For $n \in \mathbb{N}$, we define

$$g^n = \underbrace{g \cdot g \cdots g}_{n \text{ times}}$$

and

$$g^{-n} = \underbrace{g^{-1} \cdot g^{-1} \cdots g^{-1}}_{n \text{ times}}.$$

Theorem 1.8 *In a group, the usual laws of exponents hold; that is, for all $g, h \in G$,*

1. $g^m g^n = g^{m+n}$ *for all* $m, n \in \mathbb{Z}$;
2. $(g^m)^n = g^{mn}$ *for all* $m, n \in \mathbb{Z}$;
3. $(gh)^n = (h^{-1}g^{-1})^{-n}$ *for all* $n \in \mathbb{Z}$. *Furthermore, if G is abelian, then* $(gh)^n = g^n h^n$.

We will leave the proof of this theorem as an exercise. Notice that $(gh)^n \neq g^n h^n$ in general, since the group may not be abelian. If the group is $\mathbb{Z}$ or $\mathbb{Z}_n$, we write the group operation additively and the exponential operation multiplicatively; that is, we write ng instead of g^n. The laws of exponents now become

1. $mg + ng = (m + n)g$ for all $m, n \in \mathbb{Z}$;
2. $m(ng) = (mn)g$ for all $m, n \in \mathbb{Z}$;

3. $m(g+h) = mg + mh$ for all $n \in \mathbb{Z}$.

It is important to realize that the last statement can be made only because $\mathbb{Z}$ and $\mathbb{Z}_n$ are commutative groups.

Historical Note

Although the first clear axiomatic definition of a group was not given until the late 1800s, group-theoretic methods had been employed before this time in the development of many areas of mathematics, including geometry and the theory of algebraic equations.

Joseph-Louis Lagrange used group-theoretic methods in a 1770–1771 memoir to study methods of solving polynomial equations. Later, Évariste Galois (1811–1832) succeeded in developing the mathematics necessary to determine exactly which polynomial equations could be solved in terms of the polynomials' coefficients. Galois' primary tool was group theory.

The study of geometry was revolutionized in 1872 when Felix Klein proposed that geometric spaces should be studied by examining those properties that are invariant under a transformation of the space. Sophus Lie, a contemporary of Klein, used group theory to study solutions of partial differential equations. One of the first modern treatments of group theory appeared in William Burnside's *The Theory of Groups of Finite Order* [1], first published in 1897.

1.3 Subgroups

Definitions and Examples

Sometimes we wish to investigate smaller groups sitting inside a larger group. The set of even integers $2\mathbb{Z} = \{\ldots, -2, 0, 2, 4, \ldots\}$ is a group under the operation of addition. This smaller group sits naturally inside of the group of integers under addition. We define a ***subgroup*** H of a group G to be a subset H of G such that when the group operation of G is restricted to H, H is a group in its own right. Observe that every group G with at least two elements will always have at least two subgroups, the subgroup consisting of the identity element alone and the entire group itself. The subgroup $H = \{e\}$ of a group G is called the ***trivial subgroup***. A subgroup that is a proper subset of G is called a ***proper subgroup***. In many of the examples that we have investigated up to this point, there exist other subgroups besides the trivial and improper subgroups.

Example 10. Consider the set of nonzero real numbers, $\mathbb{R}^*$, with the group operation of multiplication. The identity of this group is 1 and the inverse of any element $a \in \mathbb{R}^*$ is just $1/a$. We will show that

$$\mathbb{Q}^* = \{p/q : p \text{ and } q \text{ are nonzero integers}\}$$

is a subgroup of $\mathbb{R}^*$. The identity of $\mathbb{R}^*$ is 1; however, $1 = 1/1$ is the quotient of two nonzero integers. Hence, the identity of $\mathbb{R}^*$ is in $\mathbb{Q}^*$. Given two elements in $\mathbb{Q}^*$, say p/q and r/s, their product pr/qs is also in $\mathbb{Q}^*$. The inverse of any element $p/q \in \mathbb{Q}^*$ is again in $\mathbb{Q}^*$ since $(p/q)^{-1} = q/p$. Since multiplication in $\mathbb{R}^*$ is associative, multiplication in $\mathbb{Q}^*$ is associative. ■

Example 11. Recall that $\mathbb{C}^*$ is the multiplicative group of nonzero complex numbers. Let $H = \{1, -1, i, -i\}$. Then H is a subgroup of $\mathbb{C}^*$. It is quite easy to verify that H is a group under multiplication and that $H \subset \mathbb{C}^*$. ■

Example 12. Let $SL_2(\mathbb{R})$ be the subset of $GL_2(\mathbb{R})$ consisting of matrices of determinant one; that is, a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is in $SL_2(\mathbb{R})$ exactly when $ad - bc = 1$. To show that $SL_2(\mathbb{R})$ is a subgroup of the general linear group, we must show that it is a group under matrix multiplication. The 2×2 identity matrix is in $SL_2(\mathbb{R})$, as is the inverse of the matrix A:

$$A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

It remains to show that multiplication is closed; that is, that the product of two matrices of determinant one also has determinant one. We will leave this task as an exercise. The group $SL_2(\mathbb{R})$ is called the ***special linear group***. ■

Example 13. It is important to realize that a subset H of a group G can be a group without being a subgroup of G. For H to be a subgroup of G it must inherit G's binary operation. The set of all 2×2 matrices, $\mathbb{M}_2(\mathbb{R})$, forms a group under the operation of addition. The 2×2 general linear group is a subset of $\mathbb{M}_2(\mathbb{R})$ and is a group under matrix multiplication, but it is not a subgroup of $\mathbb{M}_2(\mathbb{R})$. If we add two invertible matrices, we do not necessarily obtain another invertible matrix. Observe that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

but the zero matrix is not in $GL_2(\mathbb{R})$. ■

Example 14. One way of telling whether or not two groups are the same is by examining their subgroups. Other than the trivial subgroup and the group itself, the group $\mathbb{Z}_4$ has a single subgroup consisting of the elements 0 and 2. From the group $\mathbb{Z}_2$, we can form another group of four elements as follows. As a set this group is $\mathbb{Z}_2 \times \mathbb{Z}_2$. We perform the group operation coordinatewise; that is, $(a, b) + (c, d) = (a + c, b + d)$. Table 1.5 is an addition table for $\mathbb{Z}_2 \times \mathbb{Z}_2$. Since there are three nontrivial proper subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$, $H_1 = \{(0, 0), (0, 1)\}$, $H_2 = \{(0, 0), (1, 0)\}$, and $H_3 = \{(0, 0), (1, 1)\}$, $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ must be different groups. ■

+	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	(0,0)	(0,1)	(1,0)	(1,1)
(0,1)	(0,1)	(0,0)	(1,1)	(1,0)
(1,0)	(1,0)	(1,1)	(0,0)	(0,1)
(1,1)	(1,1)	(1,0)	(0,1)	(0,0)

Table 1.5. Addition table for $\mathbb{Z}_2 \times \mathbb{Z}_2$

Some Subgroup Theorems

Let us examine some criteria for determining exactly when a subset of a group is a subgroup.

Proposition 1.9 *A subset H of G is a subgroup if and only if it satisfies the following conditions.*

1. *The identity e of G is in H.*
2. *If $h_1, h_2 \in H$, then $h_1h_2 \in H$.*
3. *If $h \in H$, then $h^{-1} \in H$.*

Proof. First suppose that H is a subgroup of G. We must show that the three conditions hold. Since H is a group, it must have an identity e_H. We must show that $e_H = e$, where e is the identity of G. We know that $e_He_H = e_H$ and that $ee_H = e_He = e_H$; hence, $ee_H = e_He_H$. By right-hand cancellation, $e = e_H$. The second condition holds since a subgroup H is a group. To prove the third condition, let $h \in H$. Since H is a group, there is an element $h' \in H$ such that $hh' = h'h = e$. By the uniqueness of the inverse in G, $h' = h^{-1}$.

Conversely, if the three conditions hold, we must show that H is a group under the same operation as G; however, these conditions plus the associativity of the binary operation are exactly the axioms stated in the definition of a group. □

Proposition 1.10 *Let H be a subset of a group G. Then H is a subgroup of G if and only if $H \neq \emptyset$, and whenever $g, h \in H$ then gh^{-1} is in H.*

Proof. Let H be a nonempty subset of G. Then H contains some element g. So $gg^{-1} = e$ is in H. If $g \in H$, then $eg^{-1} = g^{-1}$ is also in H. Finally, let $g, h \in H$. We must show that their product is also in H. However, $g(h^{-1})^{-1} = gh \in H$. Hence, H is indeed a subgroup of G. Conversely, if g and h are in H, we want to show that $gh^{-1} \in H$. Since h is in H, its inverse h^{-1} must also be in H. Because of the closure of the group operation, $gh^{-1} \in H$. □

Exercises

1. Find all $x \in \mathbb{Z}$ satisfying each of the following equations.

   (a) $3x \equiv 2 \pmod 7$

   (b) $5x + 1 \equiv 13 \pmod{23}$

   (c) $5x + 1 \equiv 13 \pmod{26}$

   (d) $9x \equiv 3 \pmod 5$

   (e) $5x \equiv 1 \pmod 6$

   (f) $3x \equiv 1 \pmod 6$

2. Which of the following multiplication tables defined on the set $G = \{a, b, c, d\}$ form a group? Support your answer in each case.

(a)

$\circ$	a	b	c	d
a	a	c	d	a
b	b	b	c	d
c	c	d	a	b
d	d	a	b	c

(b)

$\circ$	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

(c)

$\circ$	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

(d)

$\circ$	a	b	c	d
a	a	b	c	d
b	b	a	c	d
c	c	b	a	d
d	d	d	b	c

3. Let $S = \mathbb{R} \setminus \{-1\}$ and define a binary operation on S by $a * b = a + b + ab$. Prove that $(S, *)$ is an abelian group.
4. Give an example of two elements A and B in $GL_2(\mathbb{R})$ with $AB \neq BA$.
5. Prove that the product of two matrices in $SL_2(\mathbb{R})$ has determinant one.
6. Prove that the set of matrices of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

is a group under matrix multiplication. This group, known as the ***Heisenberg group***, is important in quantum physics. Matrix multiplication in the Heisenberg group is defined by

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x' & y' \\ 0 & 1 & z' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x + x' & y + y' + xz' \\ 0 & 1 & z + z' \\ 0 & 0 & 1 \end{pmatrix}.$$

7. Prove that $\det(AB) = \det(A)\det(B)$ in $GL_2(\mathbb{R})$. Use this result to show that the binary operation in the group $GL_2(\mathbb{R})$ is closed; that is, if A and B are in $GL_2(\mathbb{R})$, then $AB \in GL_2(\mathbb{R})$.
8. Let $\mathbb{Z}_2^n = \{(a_1, a_2, \ldots, a_n) : a_i \in \mathbb{Z}_2\}$. Define a binary operation on $\mathbb{Z}_2^n$ by

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n).$$

Prove that $\mathbb{Z}_2^n$ is a group under this operation. This group is important in algebraic coding theory.
9. Show that $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ is a group under the operation of multiplication.
10. Given the groups $\mathbb{R}^*$ and $\mathbb{Z}$, let $G = \mathbb{R}^* \times \mathbb{Z}$. Define a binary operation $\circ$ on G by $(a, m) \circ (b, n) = (ab, m + n)$. Show that G is a group under this operation.
11. Let a and b be elements in a group G. Prove that $ab^n a^{-1} = (aba^{-1})^n$.
12. Let $U(n)$ be the group of units in $\mathbb{Z}_n$. If $n > 2$, prove that there is an element $k \in U(n)$ such that $k^2 = 1$ and $k \neq 1$.
13. Prove that the inverse of $g_1 g_2 \cdots g_n$ is $g_n^{-1} g_{n-1}^{-1} \cdots g_1^{-1}$.
14. Prove the remainder of Theorem 1.6: if G is a group and $a, b \in G$, then the equation $xa = b$ has unique solutions in G.
15. Prove Theorem 1.8.
16. Prove the right and left cancellation laws for a group G; that is, show that in the group G, $ba = ca$ implies $b = c$ and $ab = ac$ implies $b = c$ for elements $a, b, c \in G$.
17. Show that if $a^2 = e$ for all elements a in a group G, then G must be abelian.
18. Let $H = \{2^k : k \in \mathbb{Z}\}$. Show that H is a subgroup of $\mathbb{Q}^*$.
19. Let $n = 0, 1, 2, \ldots$ and $n\mathbb{Z} = \{nk : k \in \mathbb{Z}\}$. Prove that $n\mathbb{Z}$ is a subgroup of $\mathbb{Z}$. Show that these subgroups are the only subgroups of $\mathbb{Z}$.
20. Let $\mathbb{T} = \{z \in \mathbb{C}^* : |z| = 1\}$. Prove that $\mathbb{T}$ is a subgroup of $\mathbb{C}^*$.
21. Let G consist of the 2×2 matrices of the form

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

where $\theta \in \mathbb{R}$. Prove that G is a subgroup of $SL_2(\mathbb{R})$.

22. Prove that
$$G = \{a + b\sqrt{2} : a, b \in \mathbb{Q} \text{ and } a \text{ and } b \text{ are not both zero}\}$$
is a subgroup of $\mathbb{R}^*$ under the group operation of multiplication.

23. Let G be the group of 2×2 matrices under addition and
$$H = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a + d = 0 \right\}.$$
Prove that H is a subgroup of G.

24. Prove or disprove: $SL_2(\mathbb{Z})$, the set of 2×2 matrices with integer entries and determinant one, is a subgroup of $SL_2(\mathbb{R})$.

25. Prove that the intersection of two subgroups of a group G is also a subgroup of G.

26. Prove or disprove: If H and K are subgroups of a group G, then $H \cup K$ is a subgroup of G.

27. Prove or disprove: If H and K are subgroups of a group G, then $HK = \{hk : h \in H \text{ and } k \in K\}$ is a subgroup of G. What if G is abelian?

28. Let G be a group and $g \in G$. Show that
$$Z(G) = \{x \in G : gx = xg \text{ for all } g \in G\}$$
is a subgroup of G. This subgroup is called the ***center*** of G.

29. If $xy = x^{-1}y^{-1}$ for all x and y in G, prove that G must be abelian.

30. If $(xy)^2 = xy$ for all x and y in G, prove that G must be abelian.

31. Prove or disprove: Every nontrivial subgroup of an nonabelian group is nonabelian.

32. Let H be a subgroup of G and
$$N(H) = \{g \in G : gh = hg \text{ for all } h \in H\}.$$
Prove $N(H)$ is a subgroup of G. This subgroup is called the ***normalizer*** of H in G.

2

Permutation Groups

Permutation groups are central to the study of geometric symmetries and to Galois theory, the study of finding solutions of polynomial equations. They also provide abundant examples of nonabelian groups.

Let us recall for a moment the symmetries of the equilateral triangle $\triangle ABC$ from Chapter 1. The symmetries actually consist of permutations of the three vertices, where a ***permutation*** of the set $S = \{A, B, C\}$ is a one-to-one and onto map $\pi : S \to S$. The three vertices have the following six permutations.

$$\begin{pmatrix} A & B & C \\ A & B & C \end{pmatrix} \qquad \begin{pmatrix} A & B & C \\ C & A & B \end{pmatrix} \qquad \begin{pmatrix} A & B & C \\ B & C & A \end{pmatrix}$$
$$\begin{pmatrix} A & B & C \\ A & C & B \end{pmatrix} \qquad \begin{pmatrix} A & B & C \\ C & B & A \end{pmatrix} \qquad \begin{pmatrix} A & B & C \\ B & A & C \end{pmatrix}$$

We have used the array

$$\begin{pmatrix} A & B & C \\ B & C & A \end{pmatrix}$$

to denote the permutation that sends A to B, B to C, and C to A. That is,

$$\begin{aligned} A &\mapsto B \\ B &\mapsto C \\ C &\mapsto A. \end{aligned}$$

The symmetries of a triangle form a group. In this chapter we will study groups of this type.

2.1 Definitions and Notation

In general, the permutations of a set X form a group S_X. If X is a finite set, we can assume $X = \{1, 2, \ldots, n\}$. In this case we write S_n instead of S_X. The following theorem says that S_n is a group. We call this group the ***symmetric group on*** n ***letters***.

Theorem 2.1 *The symmetric group on n letters, S_n, is a group with $n!$ elements, where the binary operation is the composition of maps.*

Proof. The identity of S_n is just the identity map that sends 1 to 1, 2 to 2, ..., n to n. If $f : S_n \to S_n$ is a permutation, then f^{-1} exists, since f is one-to-one and onto; hence, every permutation has an inverse. Composition of maps is associative, which makes the group operation associative. We leave the proof that $|S_n| = n!$ as an exercise. $\square$

A subgroup of S_n is called a ***permutation group***.

Example 1. Consider the subgroup G of S_5 consisting of the identity permutation id and the permutations

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & 4 \end{pmatrix}$$
$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 4 & 5 \end{pmatrix}$$
$$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}.$$

The following table tells us how to multiply elements in the permutation group G.

$\circ$	id	σ	τ	μ
id	id	σ	τ	μ
σ	σ	id	μ	τ
τ	τ	μ	id	σ
μ	μ	τ	σ	id

$\blacksquare$

Remark. Though it is natural to multiply elements in a group from left to right, functions are composed from right to left. Let σ and τ be permutations on a set X. To compose σ and τ as functions, we calculate $(\sigma \circ \tau)(x) = \sigma(\tau(x))$. That is, we do τ first, then σ. There are several ways to approach this inconsistency. *We will adopt the convention of multiplying permutations right to left. To compute $\sigma\tau$, do τ first and then σ.* That is, by $\sigma\tau(x)$ we mean $\sigma(\tau(x))$. (Another way of solving this problem would be to write functions on the right; that is, instead of writing $\sigma(x)$, we could write $(x)\sigma$. We could also multiply permutations left to right to agree with the usual way of multiplying elements in a group. Certainly all of these methods have been used.

Example 2. Permutation multiplication is not usually commutative. Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$
$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}.$$

Then

$$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix},$$

but

$$\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}.$$

$\blacksquare$

Cycle Notation

The notation that we have used to represent permutations up to this point is cumbersome, to say the least. To work effectively with permutation groups, we need a more streamlined method of writing down and manipulating permutations.

A permutation $\sigma \in S_X$ is a ***cycle of length*** k if there exist elements $a_1, a_2, \ldots, a_k \in X$ such that

$$\begin{aligned} \sigma(a_1) &= a_2 \\ \sigma(a_2) &= a_3 \\ &\vdots \\ \sigma(a_k) &= a_1 \end{aligned}$$

and $\sigma(x) = x$ for all other elements $x \in X$. We will write $(a_1, a_2, \ldots, a_k)$ to denote the cycle σ. Cycles are the building blocks of all permutations.

Example 3. The permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 3 & 5 & 1 & 4 & 2 & 7 \end{pmatrix} = (162354)$$

is a cycle of length 6, whereas

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 2 & 3 & 5 & 6 \end{pmatrix} = (243)$$

is a cycle of length 3.

Not every permutation is a cycle. Consider the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix} = (1243)(56).$$

This permutation actually contains a cycle of length 2 and a cycle of length 4. ■

Example 4. It is very easy to compute products of cycles. Suppose that

$$\begin{aligned} \sigma &= (1352) \\ \tau &= (256). \end{aligned}$$

We can think of σ as

$$\begin{aligned} 1 &\mapsto 3 \\ 3 &\mapsto 5 \\ 5 &\mapsto 2 \\ 2 &\mapsto 1 \end{aligned}$$

and τ as

$$\begin{aligned} 2 &\mapsto 5 \\ 5 &\mapsto 6 \\ 6 &\mapsto 2 \end{aligned}$$

Hence, $\sigma\tau = (1356)$. If $\mu = (1634)$, then $\sigma\mu = (1652)(34)$. ■

Two cycles in S_X, $\sigma = (a_1, a_2, \ldots, a_k)$ and $\tau = (b_1, b_2, \ldots, b_l)$, are ***disjoint*** if $a_i \neq b_j$ for all i and j.

Example 5. The cycles (135) and (27) are disjoint; however, the cycles (135) and (347) are not. Calculating their products, we find that

$$\begin{aligned} (135)(27) &= (135)(27) \\ (135)(347) &= (13475). \end{aligned}$$

The product of two cycles that are not disjoint may reduce to something less complicated; the product of disjoint cycles cannot be simplified. ■

Proposition 2.2 *Let σ and τ be two disjoint cycles in S_X. Then $\sigma\tau = \tau\sigma$.*

Proof. Let $\sigma = (a_1, a_2, \ldots, a_k)$ and $\tau = (b_1, b_2, \ldots, b_l)$. We must show that $\sigma\tau(x) = \tau\sigma(x)$ for all $x \in X$. If x is neither $\{a_1, a_2, \ldots, a_k\}$ nor $\{b_1, b_2, \ldots, b_l\}$, then both σ and τ fix x. That is, $\sigma(x) = x$ and $\tau(x) = x$. Hence,

$$\sigma\tau(x) = \sigma(\tau(x)) = \sigma(x) = x = \tau(x) = \tau(\sigma(x)) = \tau\sigma(x).$$

Do not forget that we are multiplying permutations right to left, which is the opposite of the order in which we usually multiply group elements. Now suppose that $x \in \{a_1, a_2, \ldots, a_k\}$. Then $\sigma(a_i) = a_{(i \bmod k)+1}$; that is,

$$\begin{aligned} a_1 &\mapsto a_2 \\ a_2 &\mapsto a_3 \\ &\vdots \\ a_{k-1} &\mapsto a_k \\ a_k &\mapsto a_1. \end{aligned}$$

However, $\tau(a_i) = a_i$ since σ and τ are disjoint. Therefore,

$$\begin{aligned} \sigma\tau(a_i) &= \sigma(\tau(a_i)) \\ &= \sigma(a_i) \\ &= a_{(i \bmod k)+1} \\ &= \tau(a_{(i \bmod k)+1}) \\ &= \tau(\sigma(a_i)) \\ &= \tau\sigma(a_i). \end{aligned}$$

Similarly, if $x \in \{b_1, b_2, \ldots, b_l\}$, then σ and τ also commute. □

Theorem 2.3 *Every permutation in S_n can be written as the product of disjoint cycles.*

PROOF. We can assume that $X = \{1, 2, \ldots, n\}$. Let $\sigma \in S_n$, and define X_1 to be $\{\sigma(1), \sigma^2(1), \ldots\}$. The set X_1 is finite since X is finite. Now let i be the first integer in X that is not in X_1 and define X_2 by $\{\sigma(i), \sigma^2(i), \ldots\}$. Again, X_2 is a finite set. Continuing in this manner, we can define finite disjoint sets $X_3, X_4, \ldots$. Since X is a finite set, we are guaranteed that this process will end and there will be only a finite number of these sets, say r. If σ_i is the cycle defined by

$$\sigma_i(x) = \begin{cases} \sigma(x) & x \in X_i \\ x & x \notin X_i, \end{cases}$$

then $\sigma = \sigma_1 \sigma_2 \cdots \sigma_r$. Since the sets $X_1, X_2, \ldots, X_r$ are disjoint, the cycles $\sigma_1, \sigma_2, \ldots, \sigma_r$ must also be disjoint. $\square$

Example 6. Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 3 & 1 & 5 & 2 \end{pmatrix}$$
$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 1 & 5 & 6 & 4 \end{pmatrix}.$$

Using cycle notation, we can write

$$\begin{aligned} \sigma &= (1624) \\ \tau &= (13)(456) \\ \sigma\tau &= (136)(245) \\ \tau\sigma &= (143)(256). \end{aligned}$$

$\blacksquare$

Remark. From this point forward we will find it convenient to use cycle notation to represent permutations. When using cycle notation, we often denote the identity permutation by (1).

Transpositions

The simplest permutation is a cycle of length 2. Such cycles are called ***transpositions***. Since

$$(a_1, a_2, \ldots, a_n) = (a_1 a_n)(a_1 a_{n-1}) \cdots (a_1 a_3)(a_1 a_2),$$

any cycle can be written as the product of transpositions, leading to the following proposition.

Proposition 2.4 *Any permutation of a finite set containing at least two elements can be written as the product of transpositions.*

Example 7. Consider the permutation

$$(16)(253) = (16)(23)(25) = (16)(45)(23)(45)(25).$$

As we can see, there is no unique way to represent permutation as the product of transpositions. For instance, we can write the identity permutation as $(12)(12)$, as $(13)(24)(13)(24)$, and in many other ways. However, as it turns out, no permutation can be written as the product of both an even number of transpositions and an odd number of transpositions. For instance, we could represent the permutation (16) by

$$(23)(16)(23)$$

or by

$$(35)(16)(13)(16)(13)(35)(56),$$

but (16) will always be the product of an odd number of transpositions. ■

Lemma 2.5 *If the identity is written as the product of r transpositions,*

$$id = \tau_1 \tau_2 \cdots \tau_r,$$

then r is an even number.

Proof. We will employ induction on r. A transposition cannot be the identity; hence, $r > 1$. If $r = 2$, then we are done. Suppose that $r > 2$. In this case the product of the last two transpositions, $\tau_{r-1}\tau_r$, must be one of the following cases:

$$\begin{aligned}
(ab)(ab) &= id \\
(bc)(ab) &= (ac)(bc) \\
(cd)(ab) &= (ab)(cd) \\
(ac)(ab) &= (ab)(bc),
\end{aligned}$$

where a, b, c, and d are distinct.

The first equation simply says that a transposition is its own inverse. If this case occurs, delete $\tau_{r-1}\tau_r$ from the product to obtain

$$id = \tau_1 \tau_2 \cdots \tau_{r-3}\tau_{r-2}.$$

By induction $r - 2$ is even; hence, r must be even.

In each of the other three cases, we can replace $\tau_{r-1}\tau_r$ with the right-hand side of the corresponding equation to obtain a new product of r transpositions for the identity. In this new product the last occurrence of a will be in the next-to-the-last transposition. We can continue this process with $\tau_{r-2}\tau_{r-1}$ to obtain either a product of $r - 2$ transpositions or a new product of r transpositions where the last occurrence of a is in τ_{r-2}. If the identity is the product of $r - 2$ transpositions, then again we are done, by our induction hypothesis; otherwise, we will repeat the procedure with $\tau_{r-3}\tau_{r-2}$.

At some point either we will have two adjacent, identical transpositions canceling each other out or a will be shuffled so that it will appear only in the first transposition. However, the latter case cannot occur, because the identity would not fix a in this instance. Therefore, the identity permutation must be the product of $r - 2$ transpositions and, again by our induction hypothesis, we are done. □

Theorem 2.6 *If a permutation σ can be expressed as the product of an even number of transpositions, then any other product of transpositions equaling σ must also contain an even number of transpositions. Similarly, if σ can be expressed as the product of an odd number of transpositions, then any other product of transpositions equaling σ must also contain an odd number of transpositions.*

PROOF. Suppose that

$$\sigma = \sigma_1 \sigma_2 \cdots \sigma_m = \tau_1 \tau_2 \cdots \tau_n,$$

where m is even. We must show that n is also an even number. The inverse of σ^{-1} is $\sigma_m \cdots \sigma_1$. Since

$$id = \sigma \sigma_m \cdots \sigma_1 = \tau_1 \cdots \tau_n \sigma_m \cdots \sigma_1,$$

n must be even by Lemma 2.5. The proof for the case in which σ can be expressed as an odd number of transpositions is left as an exercise. $\square$

In light of Theorem 2.6, we define a permutation to be ***even*** if it can be expressed as an even number of transpositions and ***odd*** if it can be expressed as an odd number of transpositions.

The Alternating Groups

One of the most important subgroups of S_n is the set of all even permutations, A_n. The group A_n is called the ***alternating group on*** n ***letters***.

Theorem 2.7 *The set A_n is a subgroup of S_n.*

PROOF. Since the product of two even permutations must also be an even permutation, A_n is closed. The identity is an even permutation and therefore is in A_n. If σ is an even permutation, then

$$\sigma = \sigma_1 \sigma_2 \cdots \sigma_r,$$

where σ_i is a transposition and r is even. Since the inverse of any transposition is itself,

$$\sigma^{-1} = \sigma_r \sigma_{r-1} \cdots \sigma_1$$

is also in A_n. $\square$

Proposition 2.8 *The number of even permutations in S_n, $n \geq 2$, is equal to the number of odd permutations; hence, the order of A_n is $n!/2$.*

PROOF. Let A_n be the set of even permutations in S_n and B_n be the set of odd permutations. If we can show that there is a bijection between these sets, they must contain the same number of elements. Fix a transposition σ in S_n. Since $n \geq 2$, such a σ exists. Define

$$\lambda_\sigma : A_n \to B_n$$

by

$$\lambda_\sigma(\tau) = \sigma\tau.$$

Suppose that $\lambda_\sigma(\tau) = \lambda_\sigma(\mu)$. Then $\sigma\tau = \sigma\mu$ and so

$$\tau = \sigma^{-1}\sigma\tau = \sigma^{-1}\sigma\mu = \mu.$$

Therefore, λ_σ is one-to-one. We will leave the proof that λ_σ is surjective to the reader. □

Example 8. The group A_4 is the subgroup of S_4 consisting of even permutations. There are twelve elements in A_4:

(1)	(12)(34)	(13)(24)	(14)(23)
(123)	(132)	(124)	(142)
(134)	(143)	(234)	(243).

One of the end-of-chapter exercises will be to write down all the subgroups of A_4. You will find that there is no subgroup of order 6. Does this surprise you? ■

Historical Note

Lagrange first thought of permutations as functions from a set to itself, but it was Cauchy who developed the basic theorems and notation for permutations. He was the first to use cycle notation. Augustin-Louis Cauchy (1789–1857) was born in Paris at the height of the French Revolution. His family soon left Paris for the village of Arcueil to escape the Reign of Terror. One of the family's neighbors there was Pierre-Simon Laplace (1749–1827), who encouraged him to seek a career in mathematics. Cauchy began his career as a mathematician by solving a problem in geometry given to him by Lagrange. Over 800 papers were written by Cauchy on such diverse topics as differential equations, finite groups, applied mathematics, and complex analysis. He was one of the mathematicians responsible for making calculus rigorous. Perhaps more theorems and concepts in mathematics have the name Cauchy attached to them than that of any other mathematician.

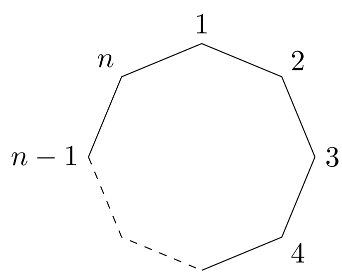


Figure 2.1. A regular n-gon

2.2 Dihedral Groups

Another special type of permutation group is the dihedral group. Recall the symmetry group of an equilateral triangle in Chapter 1. Such groups consist of the rigid motions of a regular n-sided polygon or n-gon. For $n = 3, 4, \ldots$, we define the ***nth dihedral group*** to be the group of rigid motions of a regular n-gon. We will denote this group by D_n. We can number the vertices of a regular n-gon by $1, 2, \ldots, n$ (Figure 2.1). Notice that there are exactly n choices to replace the first vertex. If we replace the first vertex by k, then the second vertex must be replaced either by vertex $k+1$ or by vertex $k-1$; hence, there are $2n$ possible rigid motions of the n-gon. We summarize these results in the following theorem.

Theorem 2.9 *The dihedral group, D_n, is a subgroup of S_n of order $2n$.*

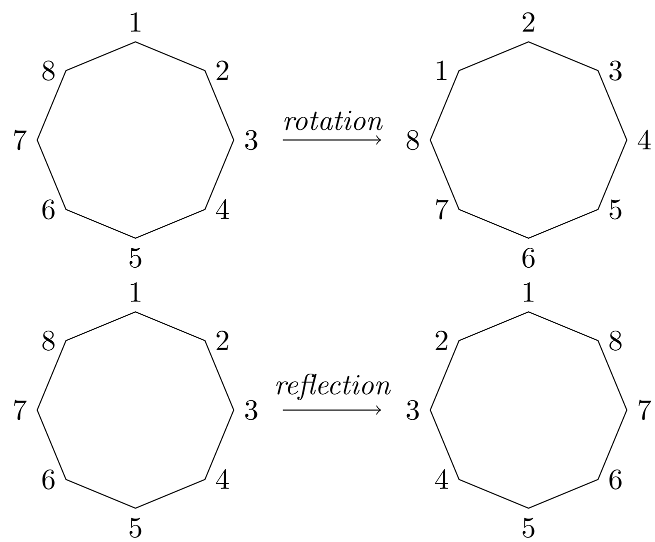


Figure 2.2. Rotations and reflections of a regular n-gon

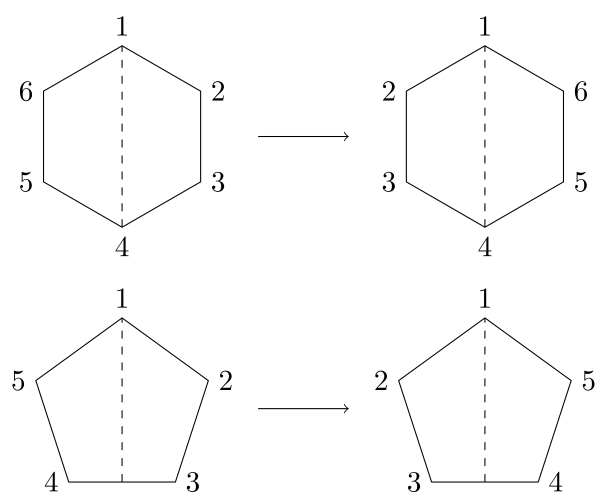


Figure 2.3. Types of reflections of a regular n-gon

Theorem 2.10 *The group D_n, $n \geq 3$, consists of all products of the two elements r and s, satisfying the relations*

$$r^n = id$$
$$s^2 = id$$
$$srs = r^{-1}.$$

Proof. The possible motions of a regular n-gon are either reflections or rotations (Figure 2.2). There are exactly n possible rotations:

$$id, \frac{360^\circ}{n}, 2 \cdot \frac{360^\circ}{n}, \ldots, (n-1) \cdot \frac{360^\circ}{n}.$$

We will denote the rotation $360^\circ/n$ by r. The rotation r generates all of the other rotations. That is,

$$r^k = k \cdot \frac{360^\circ}{n}.$$

Label the n reflections $s_1, s_2, \ldots, s_n$, where s_k is the reflection that leaves vertex k fixed. There are two cases of reflection, depending on whether n is even or odd. If there are an even number of vertices, then 2 vertices are left fixed by a reflection. If there are an odd number of vertices, then only a single vertex is left fixed by a reflection (Figure 2.3). In either case, the order of s_k is two. Let $s = s_1$. Then $s^2 = id$ and $r^n = id$. Since any rigid motion t of the n-gon replaces the first vertex by the vertex k, the second vertex must be replaced by either $k+1$ or by $k-1$. If the second vertex is replaced by $k+1$, then $t = r^{k-1}$. If it is replaced by $k-1$, then $t = r^{k-1}s$. Hence, r and s generate D_n; that is, D_n consists of all finite products of r and s. We will leave the proof that $srs = r^{-1}$ as an exercise. □

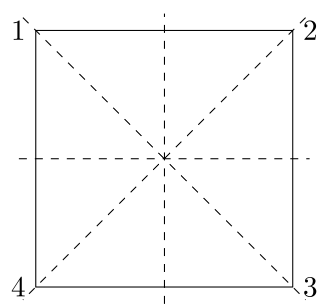


Figure 2.4. The group D_4

Example 9. The group of rigid motions of a square, D_4, consists of eight elements. With the vertices numbered 1, 2, 3, 4 (Figure 2.4), the rotations are

$$\begin{aligned} r &= (1234) \\ r^2 &= (13)(24) \\ r^3 &= (1432) \\ r^4 &= id \end{aligned}$$

and the reflections are

$$\begin{aligned} s_1 &= (24) \\ s_2 &= (13). \end{aligned}$$

The order of D_4 is 8. The remaining two elements are

$$
\begin{aligned}
rs_1 &= (12)(34) \\
r^3 s_1 &= (14)(23).
\end{aligned}
$$

■

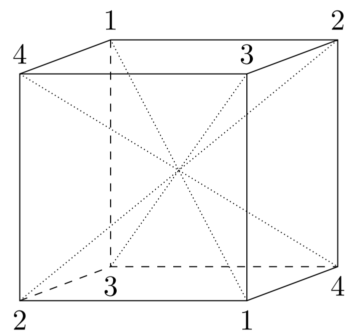


Figure 2.5. The motion group of a cube

The Motion Group of a Cube

We can investigate the groups of rigid motions of geometric objects other than a regular n-sided polygon to obtain interesting examples of permutation groups. Let us consider the group of rigid motions of a cube. One of the first questions that we can ask about this group is "what is its order?" A cube has 6 sides. If a particular side is facing upward, then there are four possible rotations of the cube that will preserve the upward-facing side. Hence, the order of the group is $6 \cdot 4 = 24$. We have just proved the following proposition.

Proposition 2.11 *The group of rigid motions of a cube contains* 24 *elements.*

Theorem 2.12 *The group of rigid motions of a cube is* S_4*.*

PROOF. From Proposition 2.11, we already know that the motion group of the cube has 24 elements, the same number of elements as there are in S_4. There are exactly four diagonals in the cube. If we label these diagonals 1, 2, 3, and 4, we must show that the motion group of the cube will give us any permutation of the diagonals (Figure 2.5). If we can obtain all of these permutations, then S_4 and the group of rigid motions of the cube must be the same. To obtain a transposition we can rotate the cube 180° about the axis joining the midpoints of opposite edges (Figure 2.6). There are six such axes, giving all transpositions in S_4. Since every element in S_4 is the product of a finite number of transpositions, the motion group of a cube must be S_4. □

Exercises

1. Write the following permutations in cycle notation.

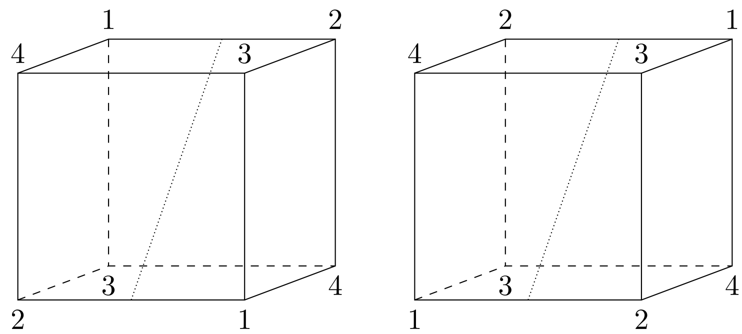


Figure 2.6. Transpositions in the motion group of a cube

(a)
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \end{pmatrix}$$

(b)
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 1 & 3 \end{pmatrix}$$

(c)
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 4 & 2 \end{pmatrix}$$

(d)
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 2 & 5 \end{pmatrix}$$

2. Compute each of the following.

(a) $(1345)(234)$
(b) $(12)(1253)$
(c) $(143)(23)(24)$
(d) $(1423)(34)(56)(1324)$
(e) $(1254)(13)(25)$
(f) $(1254)(13)(25)^2$
(g) $(1254)^{-1}(123)(45)(1254)$
(h) $(1254)^2(123)(45)$
(i) $(123)(45)(1254)^{-2}$
(j) $(1254)^{100}$
(k) $|(1254)|$
(l) $|(1254)^2|$
(m) $(12)^{-1}$
(n) $(12537)^{-1}$
(o) $[(12)(34)(12)(47)]^{-1}$
(p) $[(1235)(467)]^{-1}$

3. Express the following permutations as products of transpositions and identify them as even or odd.

(a) (14356)
(b) $(156)(234)$
(c) $(1426)(142)$
(d) $(17254)(1423)(154632)$
(e) (142637)

4. Find $(a_1, a_2, \ldots, a_n)^{-1}$.

5. Let $\sigma \in S_n$ have order n. Show that for all integers i and j, $\sigma^i = \sigma^j$ if and only if $i \equiv j \pmod{n}$.

6. Let $\sigma = \sigma_1 \cdots \sigma_m \in S_n$ be the product of disjoint cycles. Prove that the order of σ is the least common multiple of the lengths of the cycles $\sigma_1, \ldots, \sigma_m$.

7. Let $\sigma \in S_n$. Prove that σ can be written as the product of at most $n - 1$ transpositions.

8. Let $\sigma \in S_n$. If σ is not a cycle, prove that σ can be written as the product of at most $n-2$ transpositions.

9. If σ can be expressed as an odd number of transpositions, show that any other product of transpositions equaling σ must also be odd.
10. If σ is a cycle of odd length, prove that σ^2 is also a cycle.
11. Show that a 3-cycle is an even permutation.
12. Let $\tau = (a_1, a_2, \ldots, a_k)$ be a cycle of length k.
    (a) Prove that if σ is any permutation, then
    $$\sigma\tau\sigma^{-1} = (\sigma(a_1), \sigma(a_2), \ldots, \sigma(a_k))$$
    is a cycle of length k.
    (b) Let μ be a cycle of length k. Prove that there is a permutation σ such that $\sigma\tau\sigma^{-1} = \mu$.
13. For α and β in S_n, define $\alpha \sim \beta$ if there exists an $\sigma \in S_n$ such that $\sigma\alpha\sigma^{-1} = \beta$. Show that $\sim$ is an equivalence relation on S_n.

3

Cosets and Lagrange's Theorem

Lagrange's Theorem, one of the most important results in finite group theory, states that the order of a subgroup must divide the order of the group. This theorem provides a powerful tool for analyzing finite groups; it gives us an idea of exactly what type of subgroups we might expect a finite group to possess. Central to understanding Lagranges's Theorem is the notion of a coset.

3.1 Cosets

Let G be a group and H a subgroup of G. Define a ***left coset*** of H with ***representative*** $g \in G$ to be the set

$$gH = \{gh : h \in H\}.$$

Right cosets can be defined similarly by

$$Hg = \{hg : h \in H\}.$$

If left and right cosets coincide or if it is clear from the context to which type of coset that we are referring, we will use the word *coset* without specifying left or right.

Example 1. Let H be the subgroup of $\mathbb{Z}_6$ consisting of the elements 0 and 3. The cosets are

$$\begin{aligned} 0 + H = 3 + H &= \{0, 3\} \\ 1 + H = 4 + H &= \{1, 4\} \\ 2 + H = 5 + H &= \{2, 5\}. \end{aligned}$$

We will always write the cosets of subgroups of $\mathbb{Z}$ and $\mathbb{Z}_n$ with the additive notation we have used for cosets here. In a commutative group, left and right cosets are always identical. ■

Example 2. Let H be the subgroup of S_3 defined by the permutations $\{(1), (123), (132)\}$. The left cosets of H are

$$\begin{aligned} (1)H = (123)H = (132)H &= \{(1), (123), (132)\} \\ (12)H = (13)H = (23)H &= \{(12), (13), (23)\}. \end{aligned}$$

The right cosets of H are exactly the same as the left cosets:

$$H(1) = H(123) = H(132) = \{(1), (123), (132)\}$$
$$H(12) = H(13) = H(23) = \{(12), (13), (23)\}.$$

It is not always the case that a left coset is the same as a right coset. Let K be the subgroup of S_3 defined by the permutations $\{(1), (12)\}$. Then the left cosets of K are

$$(1)K = (12)K = \{(1), (12)\}$$
$$(13)K = (123)K = \{(13), (123)\}$$
$$(23)K = (132)K = \{(23), (132)\};$$

however, the right cosets of K are

$$K(1) = K(12) = \{(1), (12)\}$$
$$K(13) = K(132) = \{(13), (132)\}$$
$$K(23) = K(123) = \{(23), (123)\}.$$

■

The following lemma is quite useful when dealing with cosets. (We leave its proof as an exercise.)

Lemma 3.1 *Let H be a subgroup of a group G and suppose that $g_1, g_2 \in G$. The following conditions are equivalent.*

1. $g_1H = g_2H$;
2. $Hg_1^{-1} = Hg_2^{-1}$;
3. $g_1H \subseteq g_2H$;
4. $g_2 \in g_1H$;
5. $g_1^{-1}g_2 \in H$.

In all of our examples the cosets of a subgroup H partition the larger group G. The following theorem proclaims that this will always be the case.

Theorem 3.2 *Let H be a subgroup of a group G. Then the left cosets of H in G partition G. That is, the group G is the disjoint union of the left cosets of H in G.*

Proof. Let g_1H and g_2H be two cosets of H in G. We must show that either $g_1H \cap g_2H = \emptyset$ or $g_1H = g_2H$. Suppose that $g_1H \cap g_2H \neq \emptyset$ and $a \in g_1H \cap g_2H$. Then by the definition of a left coset, $a = g_1h_1 = g_2h_2$ for some elements h_1 and h_2 in H. Hence, $g_1 = g_2h_2h_1^{-1}$ or $g_1 \in g_2H$. By Lemma 3.1, $g_1H = g_2H$. □

Remark. There is nothing special in this theorem about left cosets. Right cosets also partition G; the proof of this fact is exactly the same as the proof for left cosets except that all group multiplications are done on the opposite side of H.

Let G be a group and H be a subgroup of G. Define the ***index*** of H in G to be the number of left cosets of H in G. We will denote the index by $[G : H]$.

Example 3. Let $G = \mathbb{Z}_6$ and $H = \{0, 3\}$. Then $[G : H] = 3$. ■

Example 4. Suppose that $G = S_3$, $H = \{(1), (123), (132)\}$, and $K = \{(1), (12)\}$. Then $[G : H] = 2$ and $[G : K] = 3$. ■

Theorem 3.3 *Let H be a subgroup of a group G. The number of left cosets of H in G is the same as the number of right cosets of H in G.*

PROOF. Let $\mathcal{L}_H$ and $\mathcal{R}_H$ denote the set of left and right cosets of H in G, respectively. If we can define a bijective map $\phi : \mathcal{L}_H \to \mathcal{R}_H$, then the theorem will be proved. If $gH \in \mathcal{L}_H$, let $\phi(gH) = Hg^{-1}$. By Lemma 3.1, the map ϕ is well-defined; that is, if $g_1H = g_2H$, then $Hg_1^{-1} = Hg_2^{-1}$. To show that ϕ is one-to-one, suppose that

$$Hg_1^{-1} = \phi(g_1H) = \phi(g_2H) = Hg_2^{-1}.$$

Again by Lemma 3.1, $g_1H = g_2H$. The map ϕ is onto since $\phi(g^{-1}H) = Hg$. □

3.2 Lagrange's Theorem

Proposition 3.4 *Let H be a subgroup of G with $g \in G$ and define a map $\phi : H \to gH$ by $\phi(h) = gh$. The map ϕ is bijective; hence, the number of elements in H is the same as the number of elements in gH.*

PROOF. We first show that the map ϕ is one-to-one. Suppose that $\phi(h_1) = \phi(h_2)$ for elements $h_1, h_2 \in H$. We must show that $h_1 = h_2$, but $\phi(h_1) = gh_1$ and $\phi(h_2) = gh_2$. So $gh_1 = gh_2$, and by left cancellation $h_1 = h_2$. To show that ϕ is onto is easy. By definition every element of gH is of the form gh for some $h \in H$ and $\phi(h) = gh$. □

Theorem 3.5 (Lagrange) *Let G be a finite group and let H be a subgroup of G. Then $|G|/|H| = [G : H]$ is the number of distinct left cosets of H in G. In particular, the number of elements in H must divide the number of elements in G.*

PROOF. The group G is partitioned into $[G : H]$ distinct left cosets. Each left coset has $|H|$ elements; therefore, $|G| = [G : H]|H|$. □

Corollary 3.6 *Suppose that G is a finite group and $g \in G$. Then the order of g must divide the number of elements in G.*

Corollary 3.7 *Let $|G| = p$ with p a prime number. Then G is cyclic and any $g \in G$ such that $g \neq e$ is a generator.*

PROOF. Let g be in G such that $g \neq e$. Then by Corollary 3.6, the order of g must divide the order of the group. Since $|\langle g \rangle| > 1$, it must be p. Hence, g generates G. □

Corollary 3.7 suggests that groups of prime order p must somehow look like $\mathbb{Z}_p$.

Corollary 3.8 *Let H and K be subgroups of a finite group G such that $G \supset H \supset K$. Then*

$$[G : K] = [G : H][H : K].$$

Proof. Observe that

$$[G : K] = \frac{|G|}{|K|} = \frac{|G|}{|H|} \cdot \frac{|H|}{|K|} = [G : H][H : K].$$

□

The converse of Lagrange's Theorem is false. The group A_4 has order 12; however, it can be shown that it does not possess a subgroup of order 6. According to Lagrange's Theorem, subgroups of a group of order 12 can have orders of either 1, 2, 3, 4, or 6. However, we are not guaranteed that subgroups of every possible order exist. To prove that A_4 has no subgroup of order 6, we will assume that it does have a subgroup H such that $|H| = 6$ and show that a contradiction must occur. The group A_4 contains eight 3-cycles; hence, H must contain a 3-cycle. We will show that if H contains one 3-cycle, then it must contain every 3-cycle, contradicting the assumption that H has only 6 elements.

Theorem 3.9 *Two cycles τ and μ in S_n have the same length if and only if there exists a $\sigma \in S_n$ such that $\mu = \sigma\tau\sigma^{-1}$.*

Proof. Suppose that

$$\begin{aligned} \tau &= (a_1, a_2, \ldots, a_k) \\ \mu &= (b_1, b_2, \ldots, b_k). \end{aligned}$$

Define σ to be the permutation

$$\begin{aligned} \sigma(a_1) &= b_1 \\ \sigma(a_2) &= b_2 \\ &\vdots \\ \sigma(a_k) &= b_k. \end{aligned}$$

Then $\mu = \sigma\tau\sigma^{-1}$.

Conversely, suppose that $\tau = (a_1, a_2, \ldots, a_k)$ is a k-cycle and $\sigma \in S_n$. If $\sigma(a_i) = b$ and $\sigma(a_{(i \bmod k)+1}) = b'$, then $\mu(b) = b'$. Hence,

$$\mu = (\sigma(a_1), \sigma(a_2), \ldots, \sigma(a_k)).$$

Since σ is one-to-one and onto, μ is a cycle of the same length as τ. □

Corollary 3.10 *The group A_4 has no subgroup of order 6.*

Proof. Since $[A_4 : H] = 2$, there are only two cosets of H in A_4. Inasmuch as one of the cosets is H itself, right and left cosets must coincide; therefore, $gH = Hg$ or $gHg^{-1} = H$ for every $g \in A_4$. By Theorem 3.9, if H contains one 3-cycle, then it must contain every 3-cycle, contradicting the order of H. □

3.3 Fermat's and Euler's Theorems

The ***Euler ϕ-function*** is the map $\phi : \mathbb{N} \to \mathbb{N}$ defined by $\phi(n) = 1$ for $n = 1$, and, for $n > 1$, $\phi(n)$ is the number of positive integers m with $1 \leq m < n$ and $\gcd(m, n) = 1$.

From Proposition 1.1, we know that the order of $U(n)$, the group of units in $\mathbb{Z}_n$, is $\phi(n)$. For example, $|U(12)| = \phi(12) = 4$ since the numbers that are relatively prime to 12 are 1, 5, 7, and 11. For any prime p, $\phi(p) = p - 1$. We state these results in the following theorem.

Theorem 3.11 *Let $U(n)$ be the group of units in $\mathbb{Z}_n$. Then $|U(n)| = \phi(n)$.*

The following theorem is an important result in number theory, due to Leonhard Euler.

Theorem 3.12 (Euler's Theorem) *Let a and n be integers such that $n > 0$ and $\gcd(a, n) = 1$. Then $a^{\phi(n)} \equiv 1 \pmod{n}$.*

PROOF. By Theorem 3.11 the order of $U(n)$ is $\phi(n)$. Consequently, $a^{\phi(n)} = 1$ for all $a \in U(n)$; or $a^{\phi(n)} - 1$ is divisible by n. Therefore, $a^{\phi(n)} \equiv 1 \pmod{n}$. □

If we consider the special case of Euler's Theorem in which $n = p$ is prime and recall that $\phi(p) = p - 1$, we obtain the following result, due to Pierre de Fermat.

Theorem 3.13 (Fermat's Little Theorem) *Let p be any prime number and suppose that $p \nmid a$. Then*

$$a^{p-1} \equiv 1 \pmod{p}.$$

Furthermore, for any integer b, $b^p \equiv b \pmod{p}$.

Historical Note

Joseph-Louis Lagrange (1736–1813), born in Turin, Italy, was of French and Italian descent. His talent for mathematics became apparent at an early age. Leonhard Euler recognized Lagrange's abilities when Lagrange, who was only 19, communicated to Euler some work that he had done in the calculus of variations. That year he was also named a professor at the Royal Artillery School in Turin. At the age of 23 he joined the Berlin Academy. Frederick the Great had written to Lagrange proclaiming that the "greatest king in Europe" should have the "greatest mathematician in Europe" at his court. For 20 years Lagrange held the position vacated by his mentor, Euler. His works include contributions to number theory, group theory, physics and mechanics, the calculus of variations, the theory of equations, and differential equations. Along with Laplace and Lavoisier, Lagrange was one of the people responsible for designing the metric system. During his life Lagrange profoundly influenced the development of mathematics, leaving much to the next generation of mathematicians in the form of examples and new problems to be solved.

Exercises

1. Suppose that G is a finite group with an element g of order 5 and an element h of order 7. Why must $|G| \geq 35$?
2. Suppose that G is a finite group with 60 elements. What are the orders of possible subgroups of G?
3. Prove or disprove: Every subgroup of the integers has finite index.

4. Prove or disprove: Every subgroup of the integers has finite order.
5. Verify Euler's Theorem for $n = 15$ and $a = 4$.
6. Use Fermat's Little Theorem to show that if $p = 4n + 3$ is prime, there is no solution to the equation $x^2 \equiv -1 \pmod{p}$.
7. Show that the integers have infinite index in the additive group of rational numbers.
8. Show that the additive group of real numbers has infinite index in the additive group of the complex numbers.
9. Let H be a subgroup of a group G and suppose that $g_1, g_2 \in G$. Prove that the following conditions are equivalent.
   (a) $g_1 H = g_2 H$
   (b) $H g_1^{-1} = H g_2^{-1}$
   (c) $g_1 H \subseteq g_2 H$
   (d) $g_2 \in g_1 H$
   (e) $g_1^{-1} g_2 \in H$
10. If $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$, show that right cosets are identical to left cosets.
11. Let G be a cyclic group of order n. Show that there are exactly $\phi(n)$ generators for G.
12. Let $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ be the factorization of n into distinct primes. Prove that
$$\phi(n) = n \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right).$$
13. Show that
$$n = \sum_{d \mid n} \phi(d)$$
for all positive integers n.

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